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Gravitational waves: from Maxwell to Einstein

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***Abstract*— The parallelism between Maxwell's electromagnetic field equations and Einstein's field equations of general relativity is exploited to reveal the equations obeyed by the recently detected gravitational waves. In the limit of weak metric perturbations, the formal solution to the gravitational wave equations is derived.**

Keywords— Einstein's field equations, gravitational waves, electromagnetic field equations

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I. Introduction

The recent excitement about the historic detection of Einstein's hundred year old prediction of gravitational waves at the Laser Interferometry Gravitational wave Observatory (LIGO) has generated renewed interest in its formal derivation from Einstein's field equations. This brief review explores the parallelism between electromagnetic theory and Einstein's theory of gravity from the field equations to the nature of the waves they support. The goal is to reveal the equations obeyed by the recently detected gravitational waves, and in the limit of weak metric perturbations, like the feeble signal detected by LIGO, the formal solution to the gravitational wave equations is deduced. With the modern reader in mind, the equations are written in MKSA to eventually motivate the geometric units.

We begin by revisiting the classical wave equation obeyed for instance by waves in stretched strings, membranes, sound waves or even bosonic relativistic particles. The inhomogeneous wave equation for the wave function Ψ is written as:

$$\left(\frac{\partial^2}{\partial(vt)^2} - \nabla^2 + k^2\right)\Psi(\mathbf{x},t) = -s(\mathbf{x},t) \qquad (1)$$

where, in Cartesian coordinates for instance, the Laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (2)$$

For mechanical waves, the speed of the wave v is determined by the medium through which the wave propagates and the characteristic wave number k. The function $s(\mathbf{x},t)$ describes the effects of the sources of the waves in the medium. Physical examples of source functions include the force driving a wave on a string, or the charge or current density in the electromagnetism.

We first review the formal structure of Maxwell's theory using 3-vector notation as we gradually progress to the 4-vector or tensor language of relativity. Then we briefly discuss Einstein's field equations and how one may exploit the parallelism between the formulations of gravity and electromagnetism to deduce gravitational wave equations.

2. Maxwell's electromagnetism

The electromagnetic field may be succinctly encoded in 2 scalar equations and 2 vector equations. This section initially presents Maxwell's equations in terms of the physical fields and then in terms of the potential fields with their gauge freedom. The wave equations are deduced and written so it conforms to the 4-vector structure they take in relativity.

2.1. Wave equation of the physical fields

It is straightforward to extract electromagnetic wave equations from Maxwell's field equations. The first noticeable complication in the field equations is the interrelation between electric **E** and magnetic **B** fields in Faraday's law and Ampere's law.

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho \qquad \text{(Gauss law for electric fields)} \qquad (3)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss law for Magnetic fields)} \qquad (4)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)} \qquad (5)$$

$$\nabla \times \mathbf{B} = \varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{J} \qquad \text{(Ampere's law)} \qquad (6)$$

$$\varepsilon_0\mu_0 = \frac{\mu_0}{4\pi k_C} = \frac{1}{c^2} \qquad (7)$$

Decoupling is achieved through the curls of Faraday's law and Ampere's law:

$$\nabla\times(\nabla\times\mathbf{E}) = -\varepsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} - \mu_0\frac{\partial\mathbf{J}}{\partial t} \qquad (8)$$

$$\nabla\times(\nabla\times\mathbf{B}) = -\varepsilon_0\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu_0\nabla\times\mathbf{J} \qquad (9)$$

Through the following identity and the Gauss laws (3) and (4),

$$\nabla\times(\nabla\times\mathbf{V}) = -\nabla^2\mathbf{V} + \nabla(\nabla\cdot\mathbf{V}) \qquad (10)$$

one finds the electromagnetic wave equations for the physical fields **E** and c**B** with derivatives of ρ and **J**/c appearing as source.

$$\left(\frac{\partial^2}{\partial(ct)^2} - \nabla^2\right)\begin{pmatrix}\mathbf{E}(\mathbf{x},t)\\ c\mathbf{B}(\mathbf{x},t)\end{pmatrix} = -4\pi k_C\begin{pmatrix}\frac{\partial}{\partial(ct)}\left(\frac{1}{c}\mathbf{J}(\mathbf{x},t)\right) + \nabla\rho(\mathbf{x},t)\\ \nabla\times\left(\frac{1}{c}\mathbf{J}(\mathbf{x},t)\right)\end{pmatrix} \qquad (11)$$

The appearance of the 4-dimensional wave operator (compare with (1)) reveals that the speed of propagation through vacuum of electromagnetic waves is exactly the speed of light c.

2.2. Electromagnetic potentials

The fact that **B** is solenoidal (4) implies that it must come from the curl of some vector potential **A**:

$$\mathbf{B} = \nabla\times\mathbf{A} \qquad (13)$$

Faraday's law then becomes:

$$\nabla\times\left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A}\right) = 0 \qquad (14)$$

This states that $\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A}$ is irrotational and may therefore be regarded as a gradient of some scalar function –φ. Thus,

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi \qquad (15)$$

Using (13), Ampere's law (6) becomes

$$\nabla\times(\nabla\times\mathbf{A}) = -\varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \varepsilon_0\mu_0\frac{\partial}{\partial t}(\nabla\varphi) + \mu_0\mathbf{J} \qquad (16)$$

Using the identity (10) and commuting space and time partial derivatives, one finds

$$\left(\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} + \nabla\left(\varepsilon_0\mu_0\frac{\partial\varphi}{\partial t} + \nabla\cdot\mathbf{A}\right) = \mu_0\mathbf{J} \qquad (17)$$

Furthermore, Gauss law (3), through (15) becomes

$$-\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} - \nabla^2\varphi = \frac{1}{\varepsilon_0}\rho \qquad (18)$$

Maxwell's field equations are now completely written in terms of the potentials φ and **A**. Notice however that although the wave operator appears in (17), it does not appear in (18). This is remedied in the next section by exploiting gauge freedom.

2.2. Gauge freedom

The physical fields (13) and (15) are invariant under the following gauge transformation generated by some scalar function λ(**x**, t):

$$\varphi \to \varphi' = \varphi - \frac{\partial\lambda}{\partial t} \qquad (19)$$

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\lambda \qquad (20)$$

Due to this gauge freedom, one can choose φ and **A** such that:

$$\nabla\cdot\mathbf{A} = \kappa - \varepsilon_0\mu_0\frac{\partial\varphi}{\partial t} \qquad (21)$$

where κ is some constant. In this gauge, equations (17) and (18) simplify to wave equations for the electromagnetic potentials (φ, c**A**) with source (ρ, **J**/c):

$$\left(\frac{\partial^2}{\partial(ct)^2} - \nabla^2\right)\begin{pmatrix}\varphi(\mathbf{x},t)\\ c\mathbf{A}(\mathbf{x},t)\end{pmatrix} = 4\pi k_C\begin{pmatrix}\rho(\mathbf{x},t)\\ \frac{1}{c}\mathbf{J}(\mathbf{x},t)\end{pmatrix} \qquad (22)$$

Note that this has a simpler source term than that of (11). We also collected together things with the same units like φ and c**A,** which are both measured in volts in MKSA. We shall see later that a gauge freedom similar to (21) is also enjoyed by gravitational waves.

2.3. Tensor notation

In relativity, time and space are considered together to form a 4-vector or a 1st-rank tensor.

$$x^\mu := \begin{pmatrix}x^0\\ x^1\\ x^2\\ x^3\end{pmatrix} = \begin{pmatrix}ct\\ x\\ y\\ z\end{pmatrix} \qquad (23)$$

This represents an event in the spacetime manifold. When the Lorentz index μ = 0, 1, 2, 3 appears as a superscript, the tensor is said to be 'contravariant' in that index. One may represent a contravariant 4-vector by a column matrix as shown in (23). While if the Lorentz index appears as a subscript, the tensor is said to be 'covariant' in that index. One may associate a row matrix to a covariant 4-vector, but with the time component negative that of the associated contravariant 4-vector.

$$x_\mu = \eta_{\mu\nu}x^\nu := \begin{pmatrix}-x^0 & x^1 & x^2 & x^3\end{pmatrix} \qquad (24)$$

As illustrated in (24), Lorentz indices are raised or lowered by the spacetime metric $g_{\mu\nu}(x)$ which, in flat spacetime, becomes the Minkowskian metric

$$\eta^{\nu\sigma} := \begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & +1 & 0 & 0\\ 0 & 0 & +1 & 0\\ 0 & 0 & 0 & +1\end{pmatrix}. \qquad (25)$$

The spacetime metric is symmetric and invertible:

$$g_{\mu\nu} = g_{\nu\mu}, \qquad g^{\sigma\rho}g_{\rho\mu} = \delta^\sigma_\mu \qquad (26)$$

Lorentz invariance

From just two postulates, Einstein developed his special theory of relativity (1905):

1. *Laws of Physics are the same in all inertial frames of reference*.
2. *The speed of light c in vacuum is the same in all inertial frames of reference.*

For instance, between two inertial observers in relative motion along their common *x*-*x*' axis (boosts along the *x* axis), the postulates lead to the Lorentz transformation equations:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & \frac{-v/c}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ \frac{-v/c}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad (27)$$

For a general direction of Lorentz boost, the transformation equations may be represented concisely in differential form:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \qquad \mu, \nu = 0,1,2,3 \quad (28)$$

Equation (28) defines how a contravariant tensor of the first rank transforms. Einstein's summation convention is employed where a sum over repeated indices is implied.

The inverse transformation is

$$dx^{\mu} = \left(\left(\frac{\partial x'}{\partial x} \right)^{-1} \right)^{\mu}_{\nu} dx'^{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} dx'^{\nu} \quad (29)$$

The Lorentz invariant square of the spacetime interval in flat spacetime is

$$c^2 d\tau^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 = -\eta_{\mu\nu} dx^{\mu} dx^{\nu} \quad (30)$$

As motivated by (22), the 4-vector potential and source may be defined as:

$$A^{\mu} := \begin{pmatrix} \varphi \\ c\mathbf{A} \end{pmatrix} = \begin{pmatrix} \varphi \\ cA_x \\ cA_y \\ cA_z \end{pmatrix} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix},$$

$$A_{\mu} := \begin{pmatrix} -\varphi & cA_x & cA_y & cA_z \end{pmatrix} = \eta_{\mu\nu} A^{\nu} \quad (31)$$

$$J^{\mu} := \begin{pmatrix} \rho \\ \frac{1}{c}\mathbf{J} \end{pmatrix} = \begin{pmatrix} \rho \\ \frac{1}{c} J_x \\ \frac{1}{c} J_y \\ \frac{1}{c} J_z \end{pmatrix} = \begin{pmatrix} J^0 \\ J^1 \\ J^2 \\ J^3 \end{pmatrix}, \quad (32)$$

The differentiations with respect to time and space may be combined together as a 4-dimensional gradient. The Maxwell field strength tensor is an anti-symmetric 2nd rank tensor defined as

$$F_{\mu\nu} \equiv \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (33)$$

where we have shown three alternative notations for the partial derivatives. Some components are evaluated for illustration:

$$F_{01} \equiv \frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1} = \frac{\partial (cA_x)}{\partial (ct)} - \frac{\partial (-\varphi)}{\partial x} = \frac{\partial A_x}{\partial t} + \frac{\partial \varphi}{\partial x} = -E_x = -F_{10}$$

$$F_{02} = \frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2} = \frac{\partial (cA_y)}{\partial (ct)} - \frac{\partial (-\varphi)}{\partial y} = \frac{\partial A_y}{\partial t} + \frac{\partial \varphi}{\partial y} = -E_y = -F_{20}$$

$$F_{12} = \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} = \frac{\partial (cA_y)}{\partial x} - \frac{\partial (cA_x)}{\partial y} = c\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = cB_z = -F_{21}$$

$$F_{13} = \frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3} = \frac{\partial (cA_z)}{\partial x} - \frac{\partial (cA_x)}{\partial z} = c\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) = -cB_y = -F_{31}$$

, etc.

Thus, the Maxwell field strength tensor has the matrix form

$$F_{\mu\nu} := \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}. \quad (34)$$

2.4 Maxwell's field equations in tensor form

Maxwell's field equations may now be concisely written as

$$F_{\mu\nu}{}^{,\nu} = 4\pi k_C J_{\mu} \quad (35)$$

In terms of the electromagnetic potentials, this becomes

$$-A_{\mu,\nu}{}^{\nu} + A_{\nu,\mu}{}^{\nu} = 4\pi k_C J_{\mu} \quad (36)$$

If we commute the partial derivatives in the second term, and introduce the gauge (21)

$$A_{\nu}{}^{,\nu} = \kappa \quad (37)$$

then we arrive at the wave equation (22) in tensor form

$$A_{\mu,\nu}{}^{\nu} = -4\pi k_C J_{\mu}. \quad (38)$$

The formal solution is written in terms of the retarded Green function

$$A_{\mu}(\mathbf{x};t) = k_C \int d^3x' \frac{J_{\mu}(\mathbf{x}'; t' = t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \quad (39)$$

3. Einstein's general relativity

3.1 Einstein's field equations

In Einstein's general theory of relativity, gravity is due to the curvature of spacetime. The metric is now a function of the spacetime $g_{\mu\nu}(x)$ and the spacetime invariant is

$$c^2 d\tau^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu \qquad (40)$$

From the extremum of the invariant

$$\delta \int_{\tau_1}^{\tau_2} d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} = 0 \qquad (41)$$

one finds the geodesic equation

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \qquad (42)$$

where the Christoffel connection is

$$\Gamma^\sigma_{\mu\nu} \equiv \frac{1}{2} g^{\sigma\rho} \left(\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right) \qquad (43)$$

The geodesic equation serves as the inertial law that governs how objects move in spacetime.

The covariant derivative or coordinate-independent derivative acting on a first rank tensor X is given by

$$\nabla_\mu X^\nu = X^\nu{}_{;\mu} = \partial_\mu X^\nu + \Gamma^\nu_{\lambda\mu} X^\lambda \qquad (44)$$

The commutator of covariant differentiation on a first rank tensor

$$\nabla_{[\mu} \nabla_{\nu]} X^\lambda = \frac{1}{2} R^\lambda{}_{\sigma\mu\nu} X^\sigma \qquad (45)$$

defines the 4th rank Riemann curvature tensor

$$R^\rho{}_{\sigma\mu\nu} \equiv \frac{\partial \Gamma^\rho_{\nu\sigma}}{\partial x^\mu} - \frac{\partial \Gamma^\rho_{\mu\sigma}}{\partial x^\nu} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \qquad (46)$$

The 2nd rank Ricci tensor is a trace of the Riemann tensor

$$R_{\mu\nu} \equiv R^\kappa{}_{\mu\kappa\nu} = \frac{\partial \Gamma^\kappa_{\nu\mu}}{\partial x^\kappa} - \frac{\partial \Gamma^\kappa_{\kappa\mu}}{\partial x^\nu} + \Gamma^\kappa_{\kappa\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\kappa_{\nu\lambda} \Gamma^\lambda_{\kappa\mu} \qquad (47)$$

$$R^\rho{}_\sigma \equiv g^{\rho\lambda} R_{\lambda\sigma} = R^\kappa{}_{\mu\kappa\mu} = \frac{\partial \Gamma^\kappa_{\mu\mu}}{\partial x^\kappa} - \frac{\partial \Gamma^\kappa_{\kappa\mu}}{\partial x^\mu} + \Gamma^\kappa_{\kappa\lambda} \Gamma^\lambda_{\mu\mu} - \Gamma^\kappa_{\mu\lambda} \Gamma^\lambda_{\kappa\mu} \qquad (48)$$

The Ricci scalar is the trace of the Ricci tensor

$$R^\rho{}_\rho \equiv g^{\rho\lambda} R_{\lambda\rho} = R^\kappa{}_{\mu\kappa\mu} = \frac{\partial \Gamma^\kappa_{\mu\mu}}{\partial x^\kappa} - \frac{\partial \Gamma^\kappa_{\kappa\mu}}{\partial x^\mu} + \Gamma^\kappa_{\kappa\lambda} \Gamma^\lambda_{\mu\mu} - \Gamma^\kappa_{\mu\lambda} \Gamma^\lambda_{\kappa\mu} \qquad (49)$$

Einstein's field equations without a cosmological constant is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (50)$$

where $T_{\mu\nu}$ acts as the energy-momentum source. A direct consequence of the Bianchi identity is the vanishing divergence of matter contributions:

$$T_{\mu\nu}{}^{;\nu} = 0 \qquad (51)$$

3.2 Weak metric perturbation

Under a weak metric perturbation

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \qquad |h_{\mu\nu}| << |\eta_{\mu\nu}| \qquad (52)$$

the linear part of Ricci Tensor may be shown to take the form

$$R^{(1)}_{\mu\nu} = \frac{1}{2} \left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} + \frac{\partial^2 h_{\mu\nu}}{\partial x_\lambda \partial x^\lambda} \right) \qquad (53)$$

Because of gauge freedom, one may choose $h_{\mu\nu}$ such that the first 3 terms vanish:

$$\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} = 0 \qquad (54)$$

Thus,

$$R^{(1)}_{\mu\nu} = \frac{1}{2} \frac{\partial^2 h_{\mu\nu}}{\partial x_\lambda \partial x^\lambda} = \frac{1}{2} \left(\frac{\partial^2}{\partial (ct)^2} - \nabla^2 \right) h_{\mu\nu} \qquad (55)$$

If all terms non-first order in h are evicted from the left hand side and absorbed in the right hand side, the field equations appear as

$$R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)} \eta_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} + \tilde{T}_{\mu\nu} \right) \qquad (56)$$

where we have defined the pseudotensor

$$\tilde{T}_{\mu\nu} \equiv \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - R^{(1)}_{\mu\nu} + \frac{1}{2} R^{(1)} \eta_{\mu\nu} \right) \qquad (57)$$

$T_{\mu\nu}$ represents the contribution from matter while $\tilde{T}_{\mu\nu}$ represents contribution from gravity.

Thus, the weak metric field perturbations obey the wave equation

$$\left(\frac{\partial^2}{\partial (ct)^2} - \nabla^2 \right) h_{\mu\nu}(x) = -\frac{16\pi G}{c^4} T_{\mu\nu} \qquad (58)$$

Finally, in analogy with (39), the solutions may be expressed as

$$h_{\mu\nu}(\mathbf{x};t) = \frac{4G}{c^4} \int d^3x' \frac{T_{\mu\nu}(\mathbf{x}'; t' = t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} \qquad (59)$$

It may be shown that gravitational waves are transverse and that they only have two degrees of freedom corresponding to 2 types of polarizations.

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